

# **SINGLE-SPECIMEN TECHNIQUE TO ESTABLISH THE J-RESISTANCE OF LINEAR VISCOELASTIC SOLIDS WITH CONSTANT POISSON'S RATIO.**

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**Abstract.** A method is developed to establish the J-resistance function for an isotropic linear viscoelastic solid of constant Poisson's ratio using the single-specimen technique with constant-rate test data. The method is based on the fact that, for a test specimen of fixed crack size under constant rate, the initiation J-integral may be established from the crack size itself, the actual external load and load-point displacement at growth initiation, and the relaxation modulus of the viscoelastic solid, without knowledge of the complete test record. Since crack size alone, of the required data, would be unknown at each point of the load-vs-load-point displacement curve of a single-specimen test, an expression is derived to estimate it. With it, the physical J-integral at each point of the test record may be established. Because of its basis on single-specimen testing, not only does the method not require the use of multiple specimens with differing initial crack sizes, but avoids the need for tracking crack growth as well.

## **INTRODUCTION**

If a master fracture-resistance function exists for a viscoelastic material, it would –most likely– depend on temperature and loading rate. If, in addition, the material has tearing ability, that resistance would depend on the amount of crack growth. In either case, any reasonably complete characterization of the fracture property function would require a large amount of data. Under such conditions, obviating costs becomes important. Such is the aim of the work in this document.

J-resistance, or more simply, J-R curves, characterize the fracture toughness property of materials that fail by ductile tearing. In principle, the J-R curve of a material, whether elastic-plastic or viscoelastic, may be constructed by testing multiple specimens having different initial crack sizes; the curve sought being the locus of collected pairs of initial crack size and J-integral at crack-growth initiation. To obviate cost, however, most J-R curve testing today utilizes the single specimen technique, which requires knowledge of the crack size corresponding to each point of the test record. Traditionally, crack sizes have been estimated either by unloading compliance measurements, or by the change in elastic potential with crack growth [3].

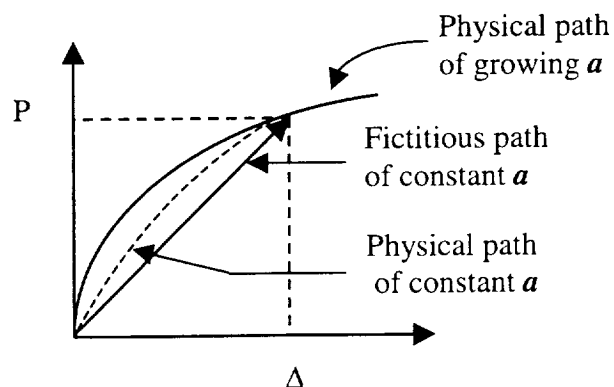
For years, researchers have been developing normalization procedures to increase the accuracy of crack-size estimates in the single-specimen technique, especially for tests at either elevated temperatures or high mechanical rates. Normalization has its origins in the work of Ernst et al [5], who showed that load-separability allows the J-R curve to be

estimated using the single-specimen technique with the load-displacement data of a test article of known initial crack size.

In its present form, normalization utilizes the actual load-vs-load point displacement record, the initial and final crack sizes, and an assumed analytical relation between crack size and displacement to estimate intermediate crack sizes. The method is being considered for adoption by several standards. In particular, a draft appendix has been prepared and reviewed for standard ASTM E-1820 [2].

Although a bounding scheme has been used recently to apply the normalization process to polymers and dynamically loaded elastic solids [6], a general approach to treat rate-dependent materials is still missing. The work here is concerned with linear isotropic viscoelastic solids with a constant Poisson's ratio. For these materials, an approach is developed to construct the J-R function from the single-specimen technique using the results of constant-rate tests. In essence, a relation is derived to estimate the crack size that corresponds to each load-displacement pair of the test record, and the means to use such information to ascertain the corresponding J-integral.

The method developed here uses the fact that, under constant-rate testing, the J-integral at crack-growth initiation for a fracture specimen of fixed crack size may be established from knowledge of the crack size itself, the values of load and load-point displacement at initiation, and the relaxation modulus of the viscoelastic material, without reference to the actual test data at that crack size. As suggested in Figure 1, the method defines a relationship that links a generic point on the load-displacement curve of a single test specimen with a growing crack to the curve that would correspond to an article of the same viscoelastic material but with a crack of fixed size, and to the secant path another test specimen of a fictitious linear elastic solid with the same crack size would follow. Using the approach of Ernst and co-workers, as well as load and J-integral separability of viscoelastic effects for the present class of materials, an equation is derived whose solution yields the crack size at each load-displacement pair.



**Figure 1.** Physical and linear elastic fictitious paths

## LOAD DEFLECTION RELATIONSHIP IN CONSTANT-RATE TESTING

The load-vs-load-point displacement response for a fracture test of a linear viscoelastic material of constant Poisson's ratio may be expressed as the product of two separate functions. One of these functions accounts for geometry and Poisson's ratio, and the other, for material heredity, in the following form [7]:

$$P(a,t) = g(a,L,\nu) \cdot \int_{0^-}^t E\left(\frac{t-\tau}{a_\tau}\right) \cdot \frac{\partial \Delta}{\partial \tau} \cdot d\tau \quad (1)$$

In this expression,  $P$  and  $\Delta$  represent, respectively, the externally applied load and load-point displacement, and the symbol  $L$  is used to denote all parameters characterizing the geometry of the test article, other than crack size,  $a$ . Also,  $E$ , is the tensile relaxation modulus of the material, and  $\nu$ , its –constant– Poisson's ratio.

If the fracture test is performed at a constant rate the load-vs-displacement relationship resembles that for a linear elastic material. This form is arrived at by direct integration of (1), after dividing and multiplying the result by  $t$ . Mathematically:

$$P(a,t) = g(a,L,\nu) \cdot \bar{E}(t) \cdot \Delta(t) \quad (2)$$

Which indeed has a linear-elastic character, except that  $\bar{E}$  represents the time-averaged value of the relaxation modulus. That is:

$$\bar{E}(t) \equiv \frac{1}{t} \cdot \int_0^t E(s) ds \quad (3)$$

## J-INTEGRAL FOR A STATIONARY CRACK IN CONSTANT RATE TESTING

Reference [7] shows the J-integral at fixed crack size for a viscoelastic solid of constant Poisson's ratio is proportional to the area under the load-vs-load displacement record of the fracture test. As it turns out, the factor of proportionality depends on crack size and other relevant geometric features of the test article, and perhaps also on the –constant– Poisson's ratio of the material at hand, but not on its relaxation modulus. Thus:

$$J = \frac{\eta(a,\nu,L)}{B_N} \cdot \int_0^\Delta P \cdot d\Delta \Big|_\Delta \quad (4)$$

This expression is similar to that in current use to estimate the plastic J-integral [4]. In it, the factor  $\eta(a,\nu,L)$ , is merely a crack-configuration function. It is related to the geometry function,  $g(a,\nu,L)$ , for a specimen of uniform thickness,  $B_N$ , in the following manner [7]:

$$\eta(a, \nu, L) \equiv -\frac{\partial[\ln(g)]}{\partial a} \quad (5)$$

For tests carried out under constant rate, the J-integral may be expressed as the product of two functions. One function depends solely on the heredity of the viscoelastic solid, and the other is the J-integral that would be obtained for a linear elastic material at the current levels of load and displacement. The latter integral is simply called the secant J-integral,  $J_s$ .

In fact, taking a constant-rate loading history  $\Delta = \dot{\Delta} \cdot t$  and the load-vs-displacement relation (2), into (4), the following is obtained, after changing integration variable and limits, and pulling out of the integral quantities independent of the integration variable.

$$J = \frac{\eta(a, \nu, L)}{B_N} \cdot g(a, \nu, L) \cdot \dot{\Delta} \cdot \dot{\Delta} \cdot \int_0^t \tau \cdot \bar{E}(\tau) \cdot d\tau$$

Multiplying and dividing the previous expression by  $t^2 \cdot \bar{E}(t)$  and re-grouping:

$$J = \frac{\eta(a, \nu, L)}{B_N} \cdot \frac{g(a, \nu, L) \cdot \bar{E}(t) \cdot \Delta}{t \cdot \bar{E}(t)} \cdot \Delta \cdot \frac{1}{t} \int_0^t \tau \cdot \bar{E}(\tau) \cdot d\tau$$

And, upon multiplying and dividing by 2, and again recalling expression (2):

$$J = \left\{ 2 \cdot \frac{\frac{1}{t} \int_0^t \tau \cdot \bar{E}(\tau) \cdot d\tau}{t \cdot \bar{E}(t)} \right\} \cdot \left[ \frac{\eta(a, \nu, L)}{B_N} \cdot \frac{1}{2} P(a, t) \cdot \Delta(t) \right] \quad (6)$$

Or, more succinctly:

$$J(a, t) = \alpha(t) \cdot J_s(a, t) \quad (7)$$

Where the function  $\alpha(t)$ , which accounts for material viscoelasticity, and the secant J-integral,  $J_s$ , corresponding to a –fictitious– linear elastic solid, have been defined as:

$$\alpha(t) \equiv 2 \cdot \frac{\frac{1}{t} \int_0^t \tau \cdot \bar{E}(\tau) \cdot d\tau}{t \cdot \bar{E}(t)} \quad (8)$$

and:

$$J_s(a,t) \equiv \frac{1}{2 \cdot B_N} \cdot \eta(a,v,L) \cdot P(a,t) \cdot \Delta(t) \quad (9)$$

The form in (7) is reminiscent –and equivalent– to the relationship between the physical J-integral, and the viscoelastic J-integral,  $J_v$ , introduced by Schapery to characterize the initiation fracture toughness of viscoelastic materials [8]. Because the secant J-integral is expressed directly in terms of physical quantities, it is easier to work with than  $J_v$ .

## J-INTEGRAL FOR A GROWING CRACK IN CONSTANT RATE TESTING

By definition, the values of  $P$  and  $\Delta$  are known at every point of the load-displacement record, irrespective of whether or not the crack size,  $a$ , is known. On the other hand, the crack-configuration function,  $\eta$ , is known –and constant– only up to the onset of crack-growth initiation. From there on, it is not known a-priori what crack size,  $a$ , is associated with any given test pair  $(\Delta, P)$ ; a fact that prevents direct use of expression (9) to estimate  $J_s$ , and hence,  $J$ . This difficulty may be circumvented by taking the total differential of (9), considered as a function of  $a$ , and  $t$ , and by integrating the resulting expression along paths of constant  $a$ , and constant  $t$ , as shown subsequently.

From (9), the total differential of  $J_s$  becomes:

$$dJ_s(a,t) = \frac{1}{2 \cdot B_N} \left[ \frac{d\eta}{da} \cdot P \cdot \Delta + \eta \cdot \frac{\partial P}{\partial a} \Delta \right] da + \frac{\eta}{2 \cdot B_N} \left[ \frac{\partial P}{\partial t} \Delta + P \dot{\Delta} \right] dt$$

The indicated partial derivatives, evaluated using (2), are:

$$\left. \frac{\partial P}{\partial a} \right|_t = \frac{dg}{da} \cdot \bar{E} \cdot \Delta \equiv \frac{1}{g} \frac{dg}{da} \cdot g \cdot \bar{E} \cdot \Delta = -\eta(a,L,v) \cdot P(a,t) \Big|_t$$

and:

$$\left. \frac{\partial P}{\partial t} \right|_a = \frac{\partial}{\partial t} \left[ g \cdot \bar{E} \cdot \dot{\Delta} \cdot t \right] \equiv g \cdot \dot{\Delta} \cdot \frac{\partial}{\partial t} \int_0^t E(s) ds = g \cdot \dot{\Delta} \cdot E(t)$$

Combining these expressions and recalling the load-deflection relationship (2), the definition of  $J_s$ , and that constant-rate testing is involved, the following results:

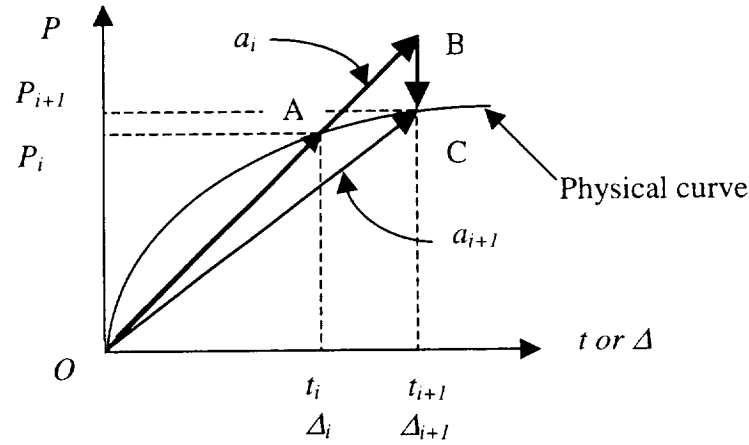
$$dJ_s(a,t) = \gamma(a,v,L) \cdot J_s(a,t) \Big|_t da + \frac{\eta \cdot g \cdot (\dot{\Delta})^2}{2B_N} \left[ t \cdot E(t) + \int_0^t E(s) ds \right] dt \quad (10)$$

In which the following definition has been introduced for convenience:

$$\gamma(a, v, L) \equiv \frac{1}{\eta(a, v, L)} \cdot \frac{d\eta}{da} - \eta(a, v, L) \quad (11)$$

Expression (10) may be used to obtain  $J_s$  at any point on the load-displacement record using any convenient load path. In particular, since under constant-rate testing  $t$  and  $\Delta$  are proportional, any generic point  $(\Delta, P)$ , or  $(t, P)$  of the test record may be reached, as suggested in Figure 2, by integrating along a path of constant crack size, followed by one at fixed time –or fixed displacement. Indeed, reaching point  $(t, P)$  along path OBC in the figure:

$$J_s(a, t) = \int_{a_i}^a \gamma(a, v, L) J_s(a, t) \Big|_t da + \frac{[\eta \cdot g]_{a_i} \left( \dot{\Delta} \right)^2}{2B_N} \left[ \int_0^t \tau \cdot E(\tau) d\tau + \int_0^t \left\{ \int_0^\tau E(s) ds \right\} d\tau \right]$$



**Figure 2.** Fictitious integration paths to adjacent points of the test record

Whose second term may be cast in a more useful form. Integrating by parts the first integral inside the brackets, collecting like terms, and, using the load-displacement relation (2), multiplying and dividing the result by  $t$ , the whole second term becomes:

$$J_s(a, t) \Big|_{a_i} \equiv J_s(a_i, t) = \frac{\eta(a_i, v, L)}{2 \cdot B_N} P(a_i, t) \cdot \Delta(t)$$

And hence:

$$J_s(a, t) = \int_{a_i}^a \gamma(a, v, L) J_s(a, t) \Big|_t da + J_s(a_i, t)$$

Or, equivalently, because time and displacement are proportional:

$$J_s(a, \Delta) = \int_{a_i}^a \gamma(a, v, L) J_s(a, \Delta) \Big|_{\Delta} da + J_s(a_i, \Delta)$$

So that, if test point  $(\Delta_{i+1}, P_{i+1})$  is reached along path  $a = a_i$ , followed by path  $\Delta = \Delta_{i+1}$ :

$$J_s(a_{i+1}, \Delta_{i+1}) = \int_{a_i}^{a_{i+1}} \gamma(a, v, L) J_s(a, \Delta_{i+1}) da + J_s(a_i, \Delta_{i+1}) \quad (12)$$

Since along a path of constant crack size the slope of the load-deflection line is  $P/\Delta$ :

$$J_s(a, \Delta_{i+1}) = \frac{\eta(a, v, L)}{2 \cdot B_N} P(a, \Delta_{i+1}) \cdot \Delta_{i+1} \equiv \frac{\eta(a, v, L)}{2 \cdot B_N} P \cdot \Delta \cdot \left( \frac{\Delta_{i+1}}{\Delta} \right)^2 \equiv J_s(a, \Delta) \cdot \left( \frac{\Delta_{i+1}}{\Delta} \right)^2$$

Which, upon multiplying and dividing by  $(\Delta_i)^2$  becomes:

$$J_s(a, \Delta_{i+1}) = J_s(a, \Delta) \cdot \left( \frac{\Delta_i}{\Delta} \right)^2 \cdot \left( \frac{\Delta_{i+1}}{\Delta_i} \right)^2 = J_s(a, \Delta_i) \cdot \left( \frac{\Delta_{i+1}}{\Delta_i} \right)^2$$

And, similarly:

$$J_s(a, \Delta_{i+1}) = J_s(a, \Delta_i) \cdot \left( \frac{\Delta_{i+1}}{\Delta_i} \right)^2 \quad (13)$$

As it should be, since the secant J-integral corresponds to a linear elastic solid, however fictitious.

With the previous results, (12) may be expressed more conveniently as:

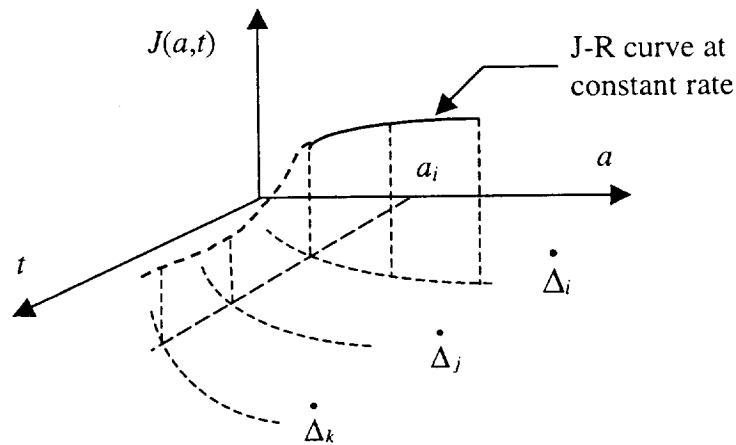
$$J_s(a_{i+1}, \Delta_{i+1}) = \int_{a_i}^{a_{i+1}} \gamma(a) \cdot J_s(a, \Delta_{i+1}) \cdot da + \left( \frac{\Delta_{i+1}}{\Delta_i} \right)^2 \cdot J_s(a_i, \Delta_i) \quad (14)$$

Note that, although  $a_i$  is known,  $a_{i+1}$  is not. However, if  $\Delta_i$  is selected close to  $\Delta_{i+1}$ ,  $a_i$  and  $a_{i+1}$  will be close together. For this reason, the integral in the previous expression may be evaluated approximately. Using the trapezoidal rule for this purpose, invoking (13) to convert  $J_s(a_i, \Delta_{i+1})$  to  $J_s(a_i, \Delta_i)$ , and (9) to express the resulting secant J-integrals in terms of their corresponding  $\eta$  functions, the following results after rearrangement:

$$\eta(a_{i+1}) = \left( \frac{1 + \frac{1}{2} \gamma(a_i) \cdot (a_{i+1} - a_i)}{1 - \frac{1}{2} \gamma(a_{i+1}) \cdot (a_{i+1} - a_i)} \right) \cdot \left( \frac{P_i / \Delta_i}{P_{i+1} / \Delta_{i+1}} \right) \cdot \eta(a_i) \quad (15)$$

Since the form of the crack configuration function,  $\eta(a)$ , may be established by analysis of the test specimen—either exact or numerical—the function  $\gamma(a)$  may be obtained from it through (11). With this, expression (15) may be solved iteratively for the crack size,  $a_{i+1}$ , corresponding to each test point  $(P_{i+1}, \Delta_{i+1})$ . This requires the triplet  $(P_o, \Delta_o, a_o)$  to be available at the outset. Incidentally, because  $a_i$  is close to  $a_{i+1}$ , the former constitutes a good initial estimate of the latter in the iteration process. Knowledge of the crack size at each point of the test record permits the secant J-integral to be estimated per (9), and the corresponding physical J-integral via (7) through (9).

The set of triplets,  $(a_i, t_i, J_i)$ , associated with each constant-rate test will define a portion of the J-resistance surface,  $J(a, t)$ , similar to that shown with the solid line in Figure 3. The collection of triplets from a sufficient number of tests carried out at different, but constant, rates will define the J-resistance surface. Clearly, if the viscoelastic material in question is thermo-rheologically simple, the time-temperature superposition may be used to stretch the time scale of the tests in the usual manner [1].



**Figure 3.** J-Resistance Surface of a Viscoelastic Solid

## CONCLUSION

The present work is concerned with isotropic linear viscoelastic solids having a constant Poisson's ratio. For such materials an approach is developed to construct the fracture resistance surface using the single-specimen technique with constant-rate tests. The basis of the proposed method is that at constant test rate and Poisson's ratio, the J-integral at crack-growth initiation for a specimen of fixed crack size is proportional to the J-integral



of a fictitious linear elastic solid at the same values of crack size, load, and displacement. In other words, it is not necessary to have knowledge of the complete load-vs-deflection curve at each crack size to obtain the corresponding initiation J-integral. The fictitious J-integral may be obtained once the crack size is known at each point on the curve. Using that the loading paths corresponding to the fictitious J-integrals are secant lines drawn from the origin, an equation is developed linking the crack size at a generic point to that of a previous point close-by. Solution of this expression yields the crack size, and hence the fictitious J-integral, and from it, the physical J-integral sought. The approach may be repeated for as many points of the test as are necessary to resolve the J-resistance curve.

The method does not require use of multiple specimens with differing initial crack sizes, and avoids the need for tracking crack growth as well. The constancy of Poisson's ratio may be relaxed using specimens whose response is independent of it—such as single-edge-notch beams— or depends on it only weakly—as with compact tension plates.

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